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The Glazman–Krein–Naimark theory for a class of discrete Hamiltonian systems ☆

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Abstract

In this paper, the Glazman–Krein–Naimark theory for a class of discrete Hamiltonian systems is developed. A minimal and a maximal operators, GKN-sets, and a boundary space for the system are introduced. Algebraic characterizations of the domains of self-adjoint extensions of the minimal operator are given. A close relationship between the domains of self-adjoint extensions and the GKN-sets is established. It is shown that there exist one-to-one correspondences among the set of all the self-adjoint extensions, the set of all the d-dimensional Lagrangian subspaces of the boundary space, and the set of all the complete Lagrangian subspaces of the boundary space.

Keywords: Discrete Hamiltonian system; The Glazman–Krein–Naimark theory; Complex symplectic geometry; Self-adjoint extension; Lagrangian subspace

1. Introduction

The Weyl–Titchmarsh theory is an important milestone in the study of spectral problems for linear ordinary differential equations. It is started with the celebrated work by H. Weyl in

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1910 [34] (or see [10,11]). He gave a dichotomy of the limit-point and limit-circle cases for singular spectral problems of formally self-adjoint second-order linear differential equations. His work was further developed for higher-order quasi-differential equations and continuous Hamiltonian systems by Titchmarsh, Kodaira, Coddington, Levinson, Atkinson, Kauffman, Read, Zettl, Hinton, Shaw, Krall, and many others (cf., e.g., [2,8,10,17–21,23,24,31,33]). So this theory is also called the Weyl–Titchmarsh theory. In recent years, investigation on spectral problems of difference operators generated by formally self-adjoint difference equations and discrete Hamiltonian systems has received a lot of attention (cf. [3–5,7,9,16,26–30]). For some other topics of discrete Hamiltonian systems, we refer [1].

Another important milestone in this area is the Glazman–Krein–Naimark (GKN) theorem. The GKN theory for ordinary linear differential operators was first established by Glazman, Krein and Naimark from 1950 onwards, and was advanced by Zettl [36]. The classical Stone–von Neumann theory provides necessary and sufficient conditions for the existence of self-adjoint extensions of closed symmetric operators in Hilbert space. As Everitt said in his report in Department Mathematisches Institut, München, Germany on November 15, 2001, the GKN theorem is equivalent to the Stone–von Neumann theory but is of particular advantage in the consideration of related boundary value problems. Everitt and Markus developed the GKN theory from the view of complex symplectic geometry (cf. [12,13]). They showed that there exists a one-to-one correspondence between the set of all the self-adjoint extensions of a minimal operator generated by quasi-differential expressions and the set of all the complete Lagrangian subspaces of a related boundary space. Recently, Zheng established the GKN theory for continuous linear Hamiltonian systems [35]. Another geometric characterization of singular self-adjoint boundary conditions for Hamiltonian systems was provided by Remling in [25]. Geometric aspects of self-adjointness in terms of complex boundary conditions for Sturm–Liouville problems were investigated in [6,22].

The above two problems have a close relation because existence of self-adjoint extensions for a given symmetric operator is of crucial importance in determining whether the related spectral problem may be employed. The purpose of this paper is to establish the GKN theory for a class of discrete Hamiltonian systems defined on a finite or an infinite interval. A complex symplectic geometric characterization is given for all the self-adjoint extensions of a minimal operator generated by the Hamiltonian system. Some ideas in the present paper are motivated by some works in [12–15,28,35].

The paper is organized as follows. In Section 2, some basic concepts about symplectic spaces and linear operators in Hilbert spaces are introduced. A minimal and a maximal operators, generated by a discrete Hamiltonian system, are defined and their properties are studied. The minimal operator is shown to be symmetric. Section 3 is devoted to the characterization of the self-adjoint extensions of the minimal operator. A boundary space and GKN-sets are introduced. A close relationship between the domains of self-adjoint extensions of the minimal operator and the GKN-sets is established. Complex symplectic geometric characterizations for all the self-adjoint extensions of the minimal operator are investigated in terms of d-dimensional Lagrangian subspaces and complete Lagrangian subspaces of the boundary space, respectively.

Remark 1.1. The GKN theory for multiple discrete Hamiltonian systems, and applications of symplectic geometry to boundary value problems for discrete Hamiltonian systems will be studied in our forthcoming papers.

2. Preliminaries

2.1. Some basic concepts

In this subsection, we collect some basic concepts about complex symplectic spaces and linear operators in Hilbert spaces, which are referred to [12,32].

Definition 2.1. A complex symplectic space S is a complex linear one, with a prescribed symplectic form $[:]: S \times S \to \mathbf{C}$, $(X, Y) \mapsto [X : Y]$ satisfying:

(1) (conjugate bilinear property) for all $X, Y, Z \in S$ and $\mu \in \mathbf{C}$,

$$[Z : X + Y] = [Z : X] + [Z : Y], \qquad [X + Y : Z] = [X : Z] + [Y : Z],$$
$$[\mu X : Y] = \mu[X : Y], \qquad [X : \mu Y] = \bar{\mu}[X : Y];$$

(2) (skew-Hermitian property) $[X : Y] = -\overline{[Y : X]}$ for all $X, Y \in S$;
(3) (non-degenerate property) $[X : Y] = 0$ for all $Y \in S$ implies that $X = 0$.

If (1) and (2) hold, then S is called a pre-symplectic space.

Definition 2.2. Let S be a complex pre-symplectic space and L be a linear subspace in S. The subspace L is called Lagrangian in case $[L : L] = 0$; that is, $[u : v] = 0$ for all $u, v \in L$. Further, a Lagrangian subspace $L \subset S$ is called complete in case $u \in S$ and $[u : L] = 0$ imply $u \in L$.

Definition 2.3. Let S_1 and S_2 be two complex symplectic spaces with symplectic forms $[:]_1$ and $[:]_2$, respectively. They are called symplectically isomorphic in case there exists a bijective linear map $h : S_1 \to S_2$ with $[hu : hv]_2 = [u : v]_1$ for all $u, v \in S_1$.

Definition 2.4. Let S be a complex symplectic space with symplectic form $[:]$, and S_1 and S_2 be two subspaces in S. S_1 and S_2 are called symplectically orthogonal in case $[u : v] = 0$ for all $u \in S_1$ and for all $v \in S_2$.

In the following, let H_1 and H_2 be two Hilbert spaces over $\mathbf{C}$ with inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$, respectively, and T be a linear operator from H_1 into H_2. Denote the domain, the range, and the kernel of the operator T by $D(T)$, $R(T)$, and $N(T)$, respectively.

Definition 2.5.

(1) T is said to be densely defined if $D(T)$ is dense in H_1.
(2) T is said to be closed if its graph $G(T) = \{(f, Tf){:}\ f \in D(T)\}$ is closed in $H_1 \times H_2$.

Assume that T is densely defined. Let

$$D^* = \{g \in H_2{:}\ \text{the functional } f \mapsto (g, Tf)_2 \text{ is continuous on } D(T)\}.$$

Since $D(T)$ is dense, then for every $g \in D^*$ there exists $h_g \in H_1$, uniquely determined by g and T via

$$(h_g, f)_1 = (g, Tf)_2 \quad \text{for all } f \in D(T).$$

The operator T^*,

$$D(T^*) := D^* \to H_1, \quad g \mapsto h_g,$$

is called the adjoint operator of T.

Definition 2.6. A linear operator T on a Hilbert space H with inner product $(\cdot,\cdot)$ is said to be Hermitian if

$$(Tf, g) = (f, Tg) \quad \text{for all } f, g \in D(T).$$

T is said to be symmetric if it is Hermitian and densely defined. Further, T is said to be self-adjoint if T is symmetric and $T = T^*$.

Let S and T be symmetric operators on Hilbert space H. If T is an extension of S, that is, $D(S) \subset D(T)$ and $Sf = Tf$ for all $f \in D(S)$, then T is said to be a symmetric extension of S. If T is an extension of S and self-adjoint, then T is said to be a self-adjoint extension of S [32]. The minimal closed symmetric extension of symmetric operator T is called its closure, and written as $\bar{T}$ [11].

2.2. Maximal and minimal operators

In this subsection, we study properties of maximal and minimal operators for the following linear discrete Hamiltonian system

$$\begin{aligned} \Delta x(t) &= A(t)x(t+1) + \big(B(t) + \lambda W_2(t)\big)u(t), \\ \Delta u(t) &= \big(C(t) - \lambda W_1(t)\big)x(t+1) - A^*(t)u(t), \end{aligned} \tag{2.1}$$

which can be written as

$$J\Delta y(t) = \big(\lambda W(t) + P(t)\big)R(y)(t) \tag*{(2.1)$'$}$$

for $t \in \mathbf{I}$, where $\mathbf{I}$ is the integer set $[a,b] := \{j\}_{j=a}^{b}$ or $[a,\infty) := \{j\}_{j=a}^{\infty}$ or $(-\infty,b] := \{j\}_{j=-\infty}^{b}$ or $\mathbf{Z}$; $R(y)(t) = (x^T(t+1), u^T(t))^T$ with $y(t) = (x^T(t), u^T(t))^T$, and $x(t), u(t) \in \mathbf{C}^n$; $A(t), B(t), C(t), W_1(t)$, and $W_2(t)$ are $n \times n$ complex-valued matrices, $A^*(t)$ is the complex conjugate transpose of $A(t)$; $B(t)$ and $C(t)$ are Hermitian matrices, $W_1(t)$ and $W_2(t)$ are both non-negative matrices;

$$P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}, \qquad W(t) = \begin{pmatrix} W_1(t) & 0 \\ 0 & W_2(t) \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

in which I_n is the $n \times n$ unit matrix.

For convenience, denote $\mathbf{I}^*$ to be $[a, b+1]$ or $[a, +\infty)$ or $(-\infty, b+1]$ or $\mathbf{Z}$ corresponding to $\mathbf{I} = [a,b]$ or $[a,+\infty)$ or $(-\infty,b] := \{\dots, b-1, b\}$ or $\mathbf{Z}$, respectively.

In this paper, we always assume that the following definiteness condition holds: for any non-trivial solution $y(t)$ of (2.1), we have

$$\sum_{t=k}^{m} R(y)^*(t)W(t)R(y)(t) > 0 \quad \forall m > k,\ k, m \in \mathbf{I}. \tag{2.2}$$

To ensure the existence and uniqueness of the solution of any initial value problem for (2.1), we always assume that $I_n - A(t)$ is non-singular on $\mathbf{I}$.

Remark 2.1. For system (2.1), the spectral problem on finite intervals was discussed in [29] and the Weyl–Titchmarsh theory over a half-line was established in [28]. Independently, Clark and Gesztesy established the Weyl–Titchmarsh theory for a class of discrete Hamiltonian systems that include system (2.1) [9].

Now we introduce the formally Hamiltonian operator l for system (2.1)

$$l(y)(t) := J\Delta y(t) - P(t)R(y)(t) \tag{2.3}$$

for $y \in D(l) := \{y\colon\ y = \{y(t)\}_{t\in\mathbf{I}^*} \subset \mathbf{C}^{2n}\}$, and the linear space

$$l^2_W(\mathbf{I}) := \left\{y \in D(l)\colon \sum_{t\in\mathbf{I}} R(y)^*(t)W(t)R(y)(t) < \infty\right\}$$

with the semi-scalar product

$$(y,z)_W = \sum_{t\in\mathbf{I}} R(z)^*(t)W(t)R(y)(t).$$

Define $\|y\|_W := (y,y)_W^{\frac{1}{2}}$ for $y \in l^2_W(\mathbf{I})$. Since $W(t)$ is merely assumed to be non-negative Hermitian matrix, $\|\cdot\|_W$ is not a norm but a semi-norm. If we define $y = z$ in $l^2_W(\mathbf{I})$ in the sense of $\|y - z\|_W = 0$, then the quotient space consisting of equivalent class of functions in $l^2_W(\mathbf{I})$ is a Hilbert space with the norm $\|\cdot\|_W$. For convenience, we denote $l^2_W(\mathbf{I})$ by the quotient space without any confusion.

Remark 2.2. For the case of $\mathbf{I} = [a,b]$ or $[a,+\infty)$, the result that $l^2_W(\mathbf{I})$ is a Hilbert space was proved in [28]. The proof is similar for the other two cases.

Define the maximal Hamiltonian operator T_1 generated by l as follows:

$$\begin{aligned} D(T_1) := \big\{y \in l^2_W(\mathbf{I})\colon &\text{ there exists } f \in l^2_W(\mathbf{I}) \text{ such that} \\ &\qquad l(y)(t) = W(t)R(f)(t),\ t \in \mathbf{I}\big\}, \\ T_1 y := f. & \end{aligned} \tag{2.4}$$

Clearly, $D(T_1)$ is a subspace of $l^2_W(\mathbf{I})$.

Lemma 2.1. *For any $f, g \in D(T_1)$ and for any $\alpha, \beta \in \mathbf{I}$ with $\alpha < \beta$,*

$$\sum_{t=\alpha}^{\beta}\big\{R(g)^*(t)W(t)R(T_1 f)(t) - R(T_1 g)^*(t)W(t)R(f)(t)\big\} = \big(g^*(t)Jf(t)\big)\Big|_{\alpha}^{\beta+1}. \tag{2.5}$$

Proof. Fix any $f, g \in D(T_1)$ and fix any $\alpha, \beta \in \mathbf{I}$ with $\alpha < \beta$. Then there exist $y, z \in l^2_W(\mathbf{I})$ such that $T_1 f = y$ and $T_1 g = z$. By Theorem 2.1 in [29], we have

$$\begin{aligned} &\sum_{t=\alpha}^{\beta}\big\{R(g)^*(t)W(t)R(T_1 f)(t) - R(T_1 g)^*(t)W(t)R(f)(t)\big\} \\ &\quad = \sum_{t=\alpha}^{\beta}\big\{R(g)^*(t)W(t)R(y)(t) - R(z)^*(t)W(t)R(f)(t)\big\} \end{aligned}$$

$$= \sum_{t=\alpha}^{\beta} \{ R(g)^*(t) l(f)(t) - l(g)^*(t) R(f)(t) \}$$
$$= \big(g^*(t) J f(t) \big) \Big|_{\alpha}^{\beta+1}.$$

This completes the proof. □

Definition 2.7. Define a boundary form [:] on $D(T_1) \times D(T_1)$ by

$$[f : g] := (T_1 f, g)_W - (f, T_1 g)_W, \quad f, g \in D(T_1). \tag{2.6}$$

It can be easily verified that the boundary form [:] is a skew-Hermitian and conjugate bilinear map from $D(T_1) \times D(T_1)$ into **C**. So, $D(T_1)$ with [:] is a pre-symplectic space.

Denote

$$D_0(T_1) := \{ y \in D(T_1)\text{: there exist } \alpha, \beta \in \mathbf{I} \text{ with } \alpha < \beta \text{ such that } y(t) = 0,\ t \in \mathbf{I}^* - (\alpha, \beta] \}.$$

Clearly, $D_0(T_1)$ is a linear subspace of $D(T_1)$. It follows from (2.5) that

$$[f : g] = 0 \quad \forall f \in D(T_1),\ \forall g \in D_0(T_1).$$

Lemma 2.2. *The subspace $D_0(T_1)$ is dense in $l_W^2(\mathbf{I})$, i.e., $\overline{D_0(T_1)} = l_W^2(\mathbf{I})$.*

Proof. The proof is the same as that of Theorem 2.4 in [28]. So we omit it here. □

Now, we define the minimal Hamiltonian operator T_0 generated by l as follows:

$$D(T_0) := \{ y \in D(T_1)\text{: } [y : D(T_1)] = 0 \}, \qquad T_0 y := T_1 y. \tag{2.7}$$

It is evident that $D(T_0)$ is a subspace of $D(T_1)$, $T_0 : D(T_0) \to l_W^2(\mathbf{I})$, and

$$D_0(T_1) \subset D(T_0) \subset D(T_1). \tag{2.8}$$

Remark 2.3. A maximal and a minimal operators for system (2.1) were given and their properties were studied in [28] in the case of $\mathbf{I} = [a, b]$ and $[a, +\infty)$. The maximal operator T_1 here is the same as the maximal operator given in [28] in these two cases. But the minimal operator T_0 here is different from that in [28], where the minimal operator is the restriction of T_1 to the domain $D_0(T_1)$. We shall discuss their relations in Theorem 2.2 below. However, it is similar to the definition of the minimal operator for differential systems [13].

Theorem 2.1. *The minimal Hamiltonian operator T_0 is symmetric.*

Proof. From the definition of T_0, we have that for all $f, g \in D(T_0)$,

$$(T_0 f, g)_W - (f, T_0 g)_W = [f : g] = 0,$$

which implies that T_0 is Hermitian. By Lemma 2.2 and from the first inclusion in (2.8), it follows that $D(T_0)$ is dense in $l_W^2(\mathbf{I})$. Hence, T_0 is symmetric. This completes the proof. □

Lemma 2.3. *Let $\{f_n\}_{n=1}^{\infty} \subset D(T_1)$ and $f \in D(T_1)$. If $f_n \to f$ and $T_1 f_n \to T_1 f$ in $l_W^2(\mathbf{I})$ as $n \to \infty$, then, for all $g \in D(T_1)$,*

$$\lim_{n\to\infty} [f_n : g] = [f : g].$$

Proof. Let $\{f_n\}_{n=1}^{\infty} \subset D(T_1)$ and $f \in D(T_1)$. Suppose that $f_n \to f$ and $T_1 f_n \to T_1 f$ in $l_W^2(\mathbf{I})$ as $n \to \infty$. Then, for any $g \in D(T_1)$, we have

$$\begin{aligned}\bigl|[f_n : g] - [f : g]\bigr| &= \bigl|[f_n - f : g]\bigr| \\ &= \bigl|\bigl(T_1(f_n - f), g\bigr)_W - (f_n - f, T_1 g)_W\bigr| \\ &\leqslant \bigl\|T_1(f_n - f)\bigr\|_W \|g\|_W + \|f_n - f\|_W \|T_1 g\|_W,\end{aligned}$$

which implies that $\lim_{n\to\infty}[f_n : g] = [f : g]$. This completes the proof. □

Theorem 2.2. *Let T_{00} be the restriction of T_1 to the domain $D_0(T_1)$. Then $T_0^* = T_{00}^* = T_1$, $T_1^* = \overline{T}_{00} = T_0$. Consequently, T_0 and T_1 are closed.*

Proof. It follows from (2.8) that

$$T_{00} \subset T_0 \subset T_1. \tag{2.9}$$

By Lemma 2.2, $D(T_{00}) = D_0(T_1)$ is dense in $l_W^2(\mathbf{I})$. So, (2.9) yields that

$$T_1^* \subset T_0^* \subset T_{00}^*. \tag{2.10}$$

We have $T_{00}^* = T_1$ by Theorem 2.5 in [28] in the case of $\mathbf{I} = [a, \infty)$ and by an argument similar to that found in the proof of Theorem 2.5 of [28] in other cases. So, to show that

$$T_0^* = T_1, \tag{2.11}$$

it suffices to show that $D(T_1) \subset D(T_0^*)$. For any fixed $f \in D(T_1)$, it follows from (2.7) that

$$[f : g] = (T_1 f, g)_W - (f, T_0 g)_W = 0 \quad \forall g \in D(T_0),$$

which implies $(f, T_0 g)_W = (T_1 f, g)_W$ for all $g \in D(T_0)$. So, $f \in D(T_0^*)$ and $T_0^* f = T_1 f$. This shows that $D(T_1) \subset D(T_0^*)$ and $T_0^* f = T_1 f$ for all $f \in D(T_1)$. Hence, (2.11) is proved. Consequently, T_1 is closed by Theorem 5.3 in [32].

Next, we show that

$$T_1^* = T_0. \tag{2.12}$$

We claim that T_0 is closed. In fact, let $\{f_n\}_{n=1}^{\infty}$ be any sequence in $D(T_0)$ satisfying that $f_n \to f$ and $T_0 f_n \to g$ in $l_W^2(\mathbf{I})$ as $n \to \infty$. It follows from (2.9) that $\{f_n\}_{n=1}^{\infty} \subset D(T_1)$ and $T_0 f_n = T_1 f_n$ for all $n \geqslant 1$. Since T_1 is closed, $f \in D(T_1)$ and $T_1 f = g$. By Lemma 2.3, we get

$$[f : h] = \lim_{n\to\infty} [f_n : h] = 0 \quad \forall h \in D(T_1),$$

which yields that $f \in D(T_0)$ and consequently, $T_0 f = T_1 f = g$. Therefore, T_0 is closed. It follows that $T_0^{**} = T_0$ again by Theorem 5.3 in [32]. So, from (2.11) we have that $T_1^* = T_0^{**} = T_0$. (2.12) is proved.

Finally, we show that $\overline{T}_{00} = T_0$. It follows from $T_{00}^* = T_1$ that T_{00}^* is densely defined. Since T_{00} is also densely defined, T_{00} is closable and $T_{00}^{**} = \overline{T}_{00}$ by Theorem 5.3 in [32]. Hence, $\overline{T}_{00} = T_{00}^{**} = T_1^* = T_0$. This completes the proof. □

Remark 2.4. The proof of Theorem 2.2 is motivated by that of Theorem 2 in [13, Appendix A], in which Everitt and Markus obtained the same result for quasi-differential expressions.

We now direct our attention to the structure of the linear subspace $D(T_1)$, which will play an important role in the study of self-adjoint extensions of T_0. Denote the deficiency spaces D^+ and D^- of T_0, respectively, by

$$\begin{aligned} D^{\pm} &:= \operatorname{span}\{f \in D(T_0^*):\ T_0^* f = \pm i f\} \\ &= \operatorname{span}\{f \in D(T_1):\ T_1 f = \pm i f\}. \end{aligned}$$

Their dimensions $d^{\pm}$ are called the positive and negative deficiency indices of T_0, respectively.

Theorem 2.3. *The sets $D(T_1)$, $D(T_0)$, D^+, and D^- satisfy the following properties*:

(1) *$D(T_1)$ is a Hilbert space with the inner product*

$$(f,g)_1 = (f,g)_W + (T_1 f, T_1 g)_W, \quad f,g \in D(T_1).$$

(2) *$D(T_0)$, D^+, and D^- are closed and pairwise orthogonal subspaces of $D(T_1)$ with inner product $(\cdot,\cdot)_1$, and satisfy*

$$D(T_1) = D(T_0) \oplus D^+ \oplus D^-. \tag{2.13}$$

(3) *$D^+ \oplus D^-$ is a complete subspace of $D(T_1)$ with inner product $(\cdot,\cdot)_1$.*
(4) *For $u \in D^+ \oplus D^-$, $(u, D^-)_1 = 0$ implies $u \in D^+$, and $(u, D^+)_1 = 0$ implies $u \in D^-$.*

Proof. The results in (3) and (4) can be easily verified by using results (1) and (2). So it suffices to show that (1) and (2) hold.

We first show that $D(T_1)$ is a Hilbert space with inner product $(\cdot,\cdot)_1$. It is evident that $(\cdot,\cdot)_1$ is an inner product defined on $D(T_1)$. So it suffices to show that $D(T_1)$ is complete in norm $\|\cdot\|_1$, defined by

$$\|f\|_1 := \left(\|f\|_W^2 + \|T_1 f\|_W^2\right)^{\frac{1}{2}}, \quad f \in D(T_1),$$

which is also called the T_1-graph norm. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $D(T_1)$ in norm $\|\cdot\|_1$. Then $\{f_n\}_{n=1}^{\infty}$ and $\{T_1 f_n\}_{n=1}^{\infty}$ are both Cauchy sequences in $l_W^2(\mathbf{I})$ in norm $\|\cdot\|_W$. Since $l_W^2(\mathbf{I})$ is complete and T_1 is closed by Theorem 2.2, there exists $f \in D(T_1)$ such that $f_n \to f$ and $T_1 f_n \to T_1 f$ in norm $\|\cdot\|_W$ as $n \to \infty$, which implies that $f_n \to f$ in norm $\|\cdot\|_1$ as $n \to \infty$. So, $D(T_1)$ with inner product $(\cdot,\cdot)_1$ is a Hilbert space.

Next, we consider (2). Clearly, $D(T_0)$, D^+, and D^- are subspaces of $D(T_1)$. By Theorem 2.2, T_0 is closed. Then $D(T_0)$ is a closed subspace of $D(T_1)$.

To show that D^+ is a closed subspace of $D(T_1)$, suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence in D^+ and converges to $f \in D(T_1)$ in norm $\|\cdot\|_1$. Then

$$T_1 f_n = i f_n, \quad n \geqslant 1,$$

which yields that $T_1 f_n \to i f$ in $\|\cdot\|_W$ as $n \to \infty$. So $T_1 f = i f$ since T_1 is closed. It follows that $f \in D^+$. Therefore, D^+ is closed.

Similarly, one can show that D^- is closed.

It is to show that $D(T_0)$, D^+, and D^- are mutually orthogonal. Fix any $u \in D(T_0)$, any $v \in D^+$, and any $w \in D^-$. Then, by Theorem 2.2, we have

$$(u,v)_1 = (u,v)_W + (T_1u, T_1v)_W = (u,v)_W + (T_0u, iv)_W$$
$$= (u,v)_W + (u, iT_1v)_W = (u,v)_W + \big(u, i^2v\big)_W = 0.$$

Similarly, one can verify that $(u,w)_1 = 0$. In addition, we have

$$(v,w)_1 = (v,w)_W + (T_1v, T_1w)_W = (v,w)_W + (iv, -iw)_W = 0.$$

Hence, $D(T_0)$, D^+, and D^- are mutually orthogonal.

To show that (2.13) holds, it suffices to show that

$$\big(D(T_0) \oplus D^+ \oplus D^-\big)^\perp = \{0\}.$$

Suppose that $\xi \in D(T_1)$ is orthogonal to $D(T_0)$, D^+, and D^-. Then

$$(\xi, u)_1 = (\xi, u)_W + (T_1\xi, T_1u)_W = 0 \quad \forall u \in D(T_0),$$

which implies that

$$(\xi, u)_W = -(T_1\xi, T_1u)_W = -(T_1\xi, T_0u)_W. \tag{2.14}$$

Since the linear functional $u \mapsto (\xi, u)_W$ is continuous on $D(T_0)$, the linear functional $u \mapsto (T_1\xi, T_0u)$ is continuous on $D(T_0)$. By Theorem 2.2, it follows that $T_1\xi \in D(T_0^*) = D(T_1)$. Thus, from (2.14) and Theorem 2.2, we get

$$(\xi, u)_W = -(T_1\xi, T_0u)_W = -\big(T_0^*(T_1\xi), u\big)_W = -\big(T_1(T_1\xi), u\big)_W,$$

which implies that

$$(\xi, f)_W = -\big(T_1(T_1\xi), f\big)_W \quad \forall f \in l_W^2(\mathbf{I}),$$

since $D(T_0)$ is dense in $l_W^2(\mathbf{I})$ by Lemma 2.2. So

$$T_1(T_1\xi) = -\xi,$$

which yields that

$$(T_1 - iI)(I - iT_1)\xi = 0.$$

Hence, $(I - iT_1)\xi \in D^+$. In addition, for any $v \in D^+$, we have that

$$0 = (\xi, v)_1 = (\xi, v)_W + (T_1\xi, T_1v)_W = (\xi, v)_W + (T_1\xi, iv)_W$$
$$= (\xi, v)_W - i(T_1\xi, v)_W = \big((I - iT_1)\xi, v\big)_W.$$

This implies $(I - iT_1)\xi = 0$ and consequently, $T_1\xi = -i\xi$, i.e., $\xi \in D^-$. Therefore, $\xi = 0$ since ξ is orthogonal to D^-.

Based on the above discussions, the proof is complete. □

3. Characterization of self-adjoint extensions of T_0

In this section, we study the complex symplectic geometric characterization of all the self-adjoint extensions of the minimal Hamiltonian operator T_0. This section is divided into four subsections.

3.1. Properties of subspaces $D(T_0)$ and $D^{\pm}$

First introduce the following important concept:

Definition 3.1. The quotient space

$$S = D(T_1)/D(T_0) \tag{3.1}$$

is called a boundary (or endpoint) space.

It follows from Theorem 2.3 that

$$\dim S = d^+ + d^-.$$

Denote the natural projection of $D(T_1)$ onto S by

$$\psi : D(T_1) \to S, \quad f \mapsto \psi f = \{f + D(T_0)\}. \tag{3.2}$$

For convenience, we also denote $\hat{f} := \psi f$ for $f \in D(T_1)$.

Based on the discussion in Section 2, $D(T_1)$ is a pre-symplectic space for the boundary form [:]. It follows from (2.7) that $D(T_0)$ is a Lagrangian subspace of $D(T_1)$. Now we define a symplectic form for the boundary space S by

$$[\hat{f} : \hat{g}] = [f : g], \quad f, g \in D(T_1). \tag{3.3}$$

Since $[f + D(T_0) : g + D(T_0)] = [f : g]$ for all $f, g \in D(T_1)$, the above form [:] is well defined on $S \times S$. In addition, because the boundary form [:] is conjugate bilinear and skew-Hermitian on $D(T_1) \times D(T_1)$, the above form [:] is also conjugate bilinear and skew-Hermitian on $S \times S$ and consequently, it is a pre-symplectic space. Further, suppose that $[\hat{k} : \hat{g}] = 0$ for some $\hat{k} \in S$ and for all $\hat{g} \in S$. Then $[k + D(T_0) : D(T_1)] = 0$ and consequently, $[k : D(T_1)] = 0$. This implies that $k \in D(T_0)$; that is, $\hat{k} = 0$. Hence, the form [:] is non-degenerate on $S \times S$ and S is a complex symplectic space.

Summing up the above discussion, we get the following result.

Proposition 3.1. *The space $D(T_1)$ is a pre-symplectic space with the boundary form* [:], *defined by* (2.6)*, $D(T_0)$ is a Lagrangian subspace of $D(T_1)$, and the boundary space S is a $(d^+ + d^-)$-dimensional complex symplectic space with the form* [:], *defined by* (3.3)*.*

In Theorem 2.3, we discussed the properties of $D(T_0)$, D^+, and D^- with respect to the inner product $(\cdot,\cdot)_1$. We now consider properties of theses sets with respect to the form [:].

Proposition 3.2. *Let* [:] *be the boundary form, defined by* (2.6)*. Then the subspaces $D(T_0)$, D^+, and D^- satisfy the following properties*:

(1) *$D(T_0)$, D^+, and D^- are pairwise symplectically orthogonal with* [:]; *that is,* $[D(T_0) : D^{\pm}] = [D^+ : D^-] = 0$;
(2) *$D^+ \oplus D^-$ is a complex symplectic space with the form* [:];
(3) *for $u \in D^+ \oplus D^-$, $[u : D^-] = 0$ implies $u \in D^+$, and $[u : D^+] = 0$ implies $u \in D^-$*;
(4) *$D^+ \oplus D^-$ and S are symplectically isomorphic.*

Proof. (1) From the definition of $D(T_0)$ in (2.7), it can be easily verified that $D(T_0)$ is symplectically orthogonal to D^+ and D^-, respectively. $[D(T_0) : D^+] = [D(T_0) : D^-] = 0$. In addition, for any $u \in D^+$ and for any $v \in D^-$, we have

$$[u : v] = (T_1 u, v)_W - (u, T_1 v)_W = (iu, v)_W - (u, -iv)_W = 0,$$

which yields that D^+ and D^- are symplectically orthogonal.

(2) It is evident that $D^+ \oplus D^-$ is a pre-symplectic space with [:]. Now, suppose that for $f \in D^+ \oplus D^-$, $[f : g] = 0$ for all $g \in D^+ \oplus D^-$. For any $h \in D(T_1)$, there exist $g_1 \in D(T_0)$ and $g_2 \in D^+ \oplus D^-$ by (2) in Theorem 2.3 such that $h = g_1 + g_2$. So,

$$[f : h] = [f : g_1 + g_2] = [f : g_1] + [f : g_2] = 0$$

by result (1) in this proposition and the assumption. Hence, $f \in D(T_0)$. Noting that $D(T_0)$ and $D^+ \oplus D^-$ are orthogonal in norm $(\cdot,\cdot)_1$ by (2) in Theorem 2.3, we have that $(f, f)_1 = 0$ and consequently, $f = 0$. Thus, [:] is non-degenerate on $D^+ \oplus D^-$ and so $D^+ \oplus D^-$ is a complex symplectic space.

(3) Suppose that $[u : D^-] = 0$ for $u = u_1 + u_2 \in D^+ \oplus D^-$, where $u_1 \in D^+$ and $u_2 \in D^-$. It follows from result (1) in this proposition that $[u_1 + u_2, D^-] = [u_2 : D^-] = 0$. So, we have

$$\begin{aligned} 0 = [u_2 : u_2] &= (T_1 u_2, u_2)_W - (u_2, T_1 u_2)_W = (-iu_2, u_2)_W - (u_2, -iu_2)_W \\ &= -2i(u_2, u_2)_W, \end{aligned}$$

which implies that $u_2 = 0$ and consequently, $u \in D^+$. Similarly, one can show that for $u \in D^+ \oplus D^-$, $[u : D^+] = 0$ implies that $u \in D^-$.

(4) Consider the natural projection map

$$\pi : D^+ \oplus D^- \to S, \quad u \mapsto \hat{u}.$$

It is evident that π is a surjective linear map. By (2) in Theorem 2.3, it can be easily verified that π is injective. Moreover, for any $u, v \in D^+ \oplus D^-$,

$$[\pi u : \pi v] = [\hat{u} : \hat{v}] = [u : v].$$

So, π is a symplectic isomorphism from $D^+ \oplus D^-$ onto S.

Summing up the above discussions, the proof is complete. □

It is known that T_0 has a self-adjoint extension if and only if $d^+ = d^-$. In this case, the dimension of S must be even. In the following, we always assume

$$\dim S = 2d. \tag{3.4}$$

Thus, T_0 has a self-adjoint extension if and only if

$$d^+ = d^- = d. \tag{3.5}$$

It follows from Theorem 5.2 in [28] that $n \leqslant d^\pm \leqslant 2n$, which yields that

$$2n \leqslant \dim S = 2d \leqslant 4n.$$

Definition 3.2. $[f : u] = 0$ is called a boundary condition if $[f : u] = 0$ for $u \in D(T_1)$ and for all $f \in D(T_1)$. A set $\{f^r\}_{r=1}^d$ in $l_W^2(\mathbf{I})$ is called a boundary condition set or a GKN-set for the pair of operators $\{T_0, T_1\}$ if

(i) $f^r \in D(T_1)$, $1 \leqslant r \leqslant d$;
(ii) $f^1, f^2, \ldots, f^d$ are linearly independent in $D(T_1)$ (modulo $D(T_0)$); that is, $\hat{f}^1, \hat{f}^2, \ldots, \hat{f}^d$ are linearly independent;
(iii) $f^1, f^2, \ldots, f^d$ are mutually symplectically orthogonal, i.e., $[f^r : f^s] = 0$, $1 \leqslant r, s \leqslant d$.

3.2. Algebraic characterizations of self-adjoint extension domains

In this subsection, we investigate algebraic characterizations of self-adjoint extension domains of T_0.

Proposition 3.3. *Assume that* (3.5) *holds. Then a linear subspace D of $l^2_W(\mathbf{I})$ is a self-adjoint extension domain of T_0 if and only if it satisfies*

(1) $D(T_0) \subset D \subset D(T_1)$;
(2) $[f : g] = 0$ *for all* $f, g \in D$;
(3) *for* $f \in D(T_1)$, $[f : g] = 0$ *for all* $g \in D$ *implies that* $f \in D$.

Proof. We first consider the necessity. Suppose that T with domain $D(T)$ is a self-adjoint extension of T_0. Then $T_0 \subset T$ and consequently, $T^* = T \subset T_0^* = T_1$ by Theorem 2.2. This implies $T_0 \subset T \subset T_1$. Hence, $D(T_0) \subset D(T) \subseteq D(T_1)$. For any $f, g \in D(T)$, we have

$$[f : g] = (T_1 f, g)_W - (f, T_1 g)_W = (Tf, g)_W - (f, Tg)_W = 0.$$

Further, suppose that $[f : g] = 0$ for $f \in D(T_1)$ and for all $g \in D(T)$. It follows that

$$0 = [f : g] = (T_1 f, g)_W - (f, T_1 g)_W = (T_1 f, g)_W - (f, Tg)_W,$$

which yields that $(T_1 f, g)_W = (f, Tg)_W$ for all $g \in D(T)$. Thus $f \in D(T^*) = D(T)$. Hence, the necessity is proved.

We now turn to consider the sufficiency. Suppose that D is a linear subspace of $l^2_W(\mathbf{I})$ and satisfies conditions (1)–(3). By Theorem 2.1, D is dense in $l^2_W(\mathbf{I})$. Define a linear operator $T : D \to l^2_W(\mathbf{I})$ by $Tf = T_1 f$ for $f \in D$. Then $T_0 \subset T \subset T_1$. It follows from (2) that $(Tf, g)_W = (f, Tg)_W$ for all $f, g \in D$. This implies that T is symmetric. Hence, $T \subseteq T^*$ and $D \subset D(T^*)$. Further, by Theorem 2.2, we have that

$$T_0 = T_1^* \subset T^* \subset T_0^* = T_1.$$

On the other hand, for any $f \in D(T^*)$, we have

$$(f, Tg)_W = (T^* f, g)_W = (T_1 f, g)_W \quad \forall g \in D,$$

which implies that $[f : g] = 0$ for all $g \in D$. So, $f \in D$ by using condition (3). Hence, $D(T^*) \subset D$ and consequently, $D = D(T^*)$ and T is self-adjoint. Therefore, the sufficiency is proved. This completes the proof. $\square$

Proposition 3.4. *Assume that* (3.5) *holds. Then a linear subspace D of $l^2_W(\mathbf{I})$ is a self-adjoint extension domain of T_0 if and only if there exists a unitary map $U : D^+ \to D^-$ such that*

$$D = \{ f \in D(T_1) \colon f = h + (I - U)\gamma,\ h \in D(T_0),\ \gamma \in D^+ \}. \tag{3.6}$$

Proof. We first consider the necessity. Suppose that T is a self-adjoint extension of T_0 with domain $D(T)$. Then the Cayley transform $U = (T-i)(T+i)^{-1}$ is a unitary transform from D^+ onto D^-. By Theorem 8.2 in [32], we have

$$T = i(I+U)(I-U)^{-1}, \qquad D(T) = \big\{ f \in D(T_1)\colon\ f = (I-U)g,\ g \in l_W^2(\mathbf{I}) \big\}.$$

So, for each $f \in D(T)$, there exists $g \in l_W^2(\mathbf{I})$ such that $f = (I-U)g$. It follows from Theorem 4.13 in [32] and Theorem 2.2 that

$$l_W^2(\mathbf{I}) = R(T_0+i) \oplus R(T_0+i)^{\perp} = R(T_0+i) \oplus N(T_1-i) = R(T_0+i) \oplus D^+.$$

Then there exist $g_1 \in R(T_0+i)$ and $g_2 \in D^+$ such that $g = g_1 + g_2$. This implies that

$$f = (I-U)g_1 + (I-U)g_2.$$

In addition, there exists $x \in D(T_0)$ such that

$$g_1 = (T_0+i)x = (T+i)x = \big(i(I+U)(I-U)^{-1} + i\big)x = 2i(I-U)^{-1}x,$$

which implies that $(I-U)g_1 = 2ix \in D(T_0)$. Thus, $D(T) \subset D$, where D is defined by (3.6). Conversely, for any $f \in D$ there exist $h \in D(T_0)$ and $\gamma \in D^+$ such that $f = h + (I-U)\gamma$. Since $D(T_0) \subset D(T)$, there exists $h_1 \in l_W^2(\mathbf{I})$ such that $h = (I-U)h_1$. So, $f = (I-U)(h_1+\gamma)$. Obviously, $h_1 + \gamma \in l_W^2(\mathbf{I})$. It follows that $D \subset D(T)$. Therefore, $D(T) = D$; that is, $D(T)$ can be represented by (3.6).

We now consider the sufficiency. Suppose that there exists a unitary map $U: D^+ \to D^-$ such that a linear subspace D of $l_W^2(\mathbf{I})$ is represented by (3.6). By Proposition 3.3, it suffices to show that D satisfies (1), (2), and (3) in Proposition 3.3.

It is clear that $D(T_0) \subset D \subset D(T_1)$ from (3.6).

Fix any $f, g \in D$. There exist $h_1, h_2 \in D(T_0)$ and $\phi_1, \phi_2 \in D^+$ such that

$$f = h_1 + (I-U)\phi_1, \qquad g = h_2 + (I-U)\phi_2.$$

Then we have

$$\begin{aligned}
[f:g] &= \big[h_1 + (I-U)\phi_1 : h_2 + (I-U)\phi_2\big] \\
&= [\phi_1:\phi_2] + [U\phi_1:U\phi_2] - [U\phi_1:\phi_2] - [\phi_1:U\phi_2].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
[\phi_1:\phi_2] &= (T_1\phi_1,\phi_2)_W - (\phi_1,T_1\phi_2)_W = 2i(\phi_1,\phi_2)_W, \\
[U\phi_1:U\phi_2] &= (T_1U\phi_1,U\phi_2)_W - (U\phi_1,T_1U\phi_2)_W \\
&= -2i(U\phi_1,U\phi_2)_W = -2i(\phi_1,\phi_2)_W,
\end{aligned}$$

and

$$[U\phi_1:\phi_2] = [\phi_1:U\phi_2] = 0$$

by Proposition 3.2. Hence, $[f:g] = 0$.

Further, suppose that $[f:g] = 0$ for $f \in D(T_1)$ and for all $g \in D$. There exist $h_1 \in D(T_0)$, $f_1 \in D^+$, and $f_2 \in D^-$ such that $f = h_1 + f_1 + f_2$ by Theorem 2.3, and there exist $h_2 \in D(T_0)$ and $g_1 \in D^+$ such that $g = h_2 + (I-U)g_1$. It follows from Proposition 3.2 that

$$[f_1:Ug_1] = [f_2:g_1] = 0.$$

In addition, we have

$$[f_1 : g_1] = (T_1 f_1, g_1)_W - (f_1, T_1 g_1)_W = 2i(f_1, g_1)_W,$$
$$[f_2 : Ug_1] = (T_1 f_2, Ug_1)_W - (f_2, T_1 Ug_1)_W = -2i(f_2, Ug_1)_W.$$

Then, the above relations imply that

$$0 = [f : g] = \big[h_1 + f_1 + f_2 : h_2 + (I - U)g_1\big] = 2i\big((f_1, g_1)_W + (f_2, Ug_1)_W\big),$$

which yields that $(f_1, g_1)_W = -(f_2, Ug_1)_W$. Consequently, we get that $(Uf_1, Ug_1)_W = (f_1, g_1)_W = -(f_2, Ug_1)_W$. Since U is unitary and g_1 can be arbitrarily chosen in D^+, there exists $g_1 \in D^+$ such that $Ug_1 = Uf_1 + f_2$. Hence, $Uf_1 + f_2 = 0$, i.e., $f_2 = -Uf_1$ and consequently, $f \in D$. Therefore, D satisfies (1)–(3) in Proposition 3.3. This completes the proof. □

The following result is a direct consequence of Proposition 3.4.

Corollary 3.1. *Assume that* (3.5) *holds. Then a linear subspace D of $l^2_W(\mathbf{I})$ is a self-adjoint extension domain of T_0 if and only if there exists a unitary map $U : D^+ \to D^-$ such that*

$$D = \left\{ f \in D(T_1):\ f = h + \sum_{j=1}^{d} \alpha_j (I - U)\gamma_j,\ h \in D(T_0),\ \alpha_j \in \mathbf{C},\ 1 \leqslant j \leqslant d \right\}, \tag{3.7}$$

where $\{\gamma_j\}_{j=1}^d$ is an orthonormal basis of D^+.

Proposition 3.5. *Assume that* (3.5) *holds. Then a linear subspace D of $l^2_W(\mathbf{I})$ is a self-adjoint extension domain of T_0 if and only if there exists a unitary map $U : D^+ \to D^-$ such that*

$$D = \big\{ f \in D(T_1):\ [f : \Psi_j] = 0,\ 1 \leqslant j \leqslant d \big\}, \tag{3.8}$$

where

$$\Psi_j = (I - U)\gamma_j, \quad 1 \leqslant j \leqslant d, \tag{3.9}$$

and $\{\gamma_j\}_{j=1}^d$ is an orthonormal basis of D^+.

Proof. It suffices to show that the two sets defined by (3.7) and (3.8), respectively, are equal for the same unitary map $U : D^+ \to D^-$ by Corollary 3.1. For convenience, denote the set D in (3.7) by D_1 and the set D in (3.8) by D_2. It is to show $D_1 = D_2$.

For any $f \in D_1$, there exist $h \in D(T_0)$, $\alpha_j \in \mathbf{C}$, $1 \leqslant j \leqslant d$, such that

$$f = h + \sum_{j=1}^{d} \alpha_j \Psi_j,$$

where Ψ_j is defined by (3.9). It can be easily verified that

$$[\gamma_j : \gamma_k] = -[U\gamma_j : U\gamma_k] = 2i\delta_{jk}, \quad [\gamma_j : U\gamma_k] = 0, \quad j, k = 1, 2, \ldots, d, \tag{3.10}$$

by Proposition 3.2 and by using the fact that U is a unitary transform from D^+ onto D^-. This implies that $[\Psi_r : \Psi_k] = 0$, $1 \leqslant r, k \leqslant d$. So, we have that for $k = 1, 2, \ldots, d$,

$$[f : \Psi_k] = [h : \Psi_k] + \left[\sum_{r=1}^{d} \alpha_r \Psi_r : \Psi_k \right] = \sum_{r=1}^{d} \alpha_r [\Psi_r : \Psi_k] = 0,$$

which yields $f \in D_2$. Thus $D_1 \subset D_2$.

Now we consider the inverse inclusion relation. Fix any $f \in D_2$. Then, $[f : \Psi_k] = 0$, $1 \leqslant k \leqslant d$, which, together with (3.9), implies that $[f : g] = 0$ for all $g \in D_1$. It follows from (3) in Proposition 3.3 that $f \in D_1$ and consequently, $D_2 \subset D_1$. Hence, $D_1 = D_2$. This completes the proof. □

3.3. Relationships between self-adjoint extension domains and GKN-sets

In this subsection, we study relationships between domains of self-adjoint extensions of T_0 and GKN-sets for $\{T_0, T_1\}$.

Lemma 3.1. *Assume that* (3.5) *holds. Let U be the unitary map from D^+ onto D^- in Proposition* 3.5 *and $\{\Psi_j\}_{j=1}^d$ be determined by* (3.9) *for some orthonormal basis $\{\gamma_j\}_{j=1}^d$ of D^+. Then $\{\Psi_j\}_{j=1}^d$ is a GKN-set for $\{T_0, T_1\}$.*

Proof. It follows from the proof of Proposition 3.5 that $[\Psi_j : \Psi_k] = 0$, $1 \leqslant j, k \leqslant d$. Clearly, $\Psi_j \in D^+ \oplus D^- \subset D(T_1)$ for $1 \leqslant j \leqslant d$. By Definition 3.2, it suffices to show that $\Psi_1, \Psi_2, \ldots,$ and Ψ_d are linearly independent in $D(T_1)$ modulo $D(T_0)$. Suppose that there are constants $c_j \in \mathbf{C}$, $1 \leqslant j \leqslant d$, such that

$$c_1\Psi_1 + c_2\Psi_2 + \cdots + c_d\Psi_d = 0 \quad \big(\text{modulo } D(T_0)\big),$$

which, together with (3.9), implies

$$\sum_{j=1}^{d} c_j\gamma_j = \sum_{j=1}^{d} c_j U\gamma_j \quad \big(\text{modulo } D(T_0)\big).$$

So, it follows from (3.10) that

$$2ic_r = \left[\sum_{j=1}^{d} c_j\gamma_j : \gamma_r\right] = \left[\sum_{j=1}^{d} c_j U\gamma_j : \gamma_r\right] = 0, \quad 1 \leqslant r \leqslant d.$$

Hence, $c_r = 0$, $1 \leqslant r \leqslant d$, and consequently, $\Psi_1, \Psi_2, \ldots, \Psi_d$ are linearly independent in $D(T_1)$ modulo $D(T_0)$. The proof is complete. □

Lemma 3.2. *Assume that* (3.5) *holds and that $\{\beta_j\}_{j=1}^d$ is a GKN-set for $\{T_0, T_1\}$. Then*

$$\big\{f \in D(T_1)\colon\ [f : \beta_j] = 0,\ 1 \leqslant j \leqslant d\big\} = \operatorname{span}\{\beta_j\colon\ 1 \leqslant j \leqslant d\} + D(T_0), \tag{3.11}$$

the dimension of which modulo $D(T_0)$ is equal to d.

Proof. Suppose that $\{\beta_j\}_{j=1}^d$ is a GKN-set for $\{T_0, T_1\}$. For convenience, denote

$$D = \big\{f \in D(T_1)\colon\ [f : \beta_j] = 0,\ 1 \leqslant j \leqslant d\big\},$$
$$D' = \operatorname{span}\{\beta_j\colon\ 1 \leqslant j \leqslant d\} + D(T_0) \tag{3.12}$$

and

$$\hat{D} = \big\{\hat{f} \in S\colon\ [\hat{f} : \hat{\beta}_j] = 0,\ 1 \leqslant j \leqslant d\big\},$$
$$\hat{D}' = \operatorname{span}\{\hat{\beta}_j\colon\ 1 \leqslant j \leqslant d\}. \tag{3.13}$$

It is evident that (3.11) holds, i.e., $D = D'$, if and only if

$$\hat{D} = \hat{D}'. \tag{3.14}$$

It follows from (3.13) that $\hat{D}$ and $\hat{D}'$ are both subspaces of S with $\hat{D}' \subset \hat{D}$ and $\dim \hat{D}' = d$. So, $\dim \hat{D} \geqslant d$. In order to show that (3.14) holds, it suffices to show $\dim \hat{D} = d$. If it is not true, then $\dim \hat{D} > d$, i.e., $\dim D/D(T_0) > d$. Since $\dim D(T_1)/D(T_0) = 2d$, it follows that

$$D(T_1)/D \leqslant d - 1.$$

Introduce the following subspaces of $D(T_1)$:

$$D_0 := D(T_1), \quad D_i := \big\{f \in D(T_1)\colon [f : \beta_j] = 0,\ 1 \leqslant j \leqslant i\big\}, \quad 1 \leqslant i \leqslant d.$$

Then

$$D = D_d \subset \cdots \subset D_1 \subset D_0.$$

It follows that there exists i, $1 \leqslant i \leqslant d$, such that $D_{i-1} = D_i$. This means that if $f \in D(T_1)$ satisfies $[f : \beta_j] = 0$ for $j = 1, 2, \ldots, i-1$, then $[f : \beta_i] = 0$. Now, define linear functionals υ_j on $D(T_1)$ by

$$\upsilon_j(f) = [f : \beta_j], \quad 1 \leqslant j \leqslant d.$$

Then

$$\bigcap_{j=1}^{i-1} \big\{f \in D(T_1)\colon \upsilon_j(f) = 0\big\} \subset \big\{f \in D(T_1)\colon \upsilon_i(f) = 0\big\}.$$

By Theorem 4.1 in [32], there exist constants $c_j \in \mathbf{C}$, $1 \leqslant j \leqslant i-1$, such that

$$\upsilon_i = \sum_{j=1}^{i-1} c_j \upsilon_j,$$

which yields that

$$[f : \beta_i] = \sum_{j=1}^{i-1} c_j [f : \beta_j] \quad \forall f \in D(T_1).$$

This implies that $\beta_i - \sum_{j=1}^{i-1} c_j \beta_j \in D(T_0)$, which contradicts the assumption that $\hat{\beta}_1, \ldots, \hat{\beta}_d$ are linearly independent. So, $\dim \hat{D} = d$ and consequently, (3.14) holds. This completes the proof. □

Proposition 3.6. *Assume that* (3.5) *holds and the set* $\{\beta_j\}_{j=1}^d$ *is a GKN-set for* $\{T_0, T_1\}$. *Let* D *be defined by* (3.12). *Then* D *is the domain of a self-adjoint extension* T *of* T_0, *with* $Tf := T_1 f$, $f \in D(T) := D$.

Proof. It suffices to show that D satisfies conditions (1)–(3) in Proposition 3.3.

It is clear that $D(T_0) \subset D \subset D(T_1)$. So condition (1) in Proposition 3.3 holds.

Fix any $f, g \in D$. By Lemma 3.2, $g \in D'$. Then there exist constants $\alpha_j \in \mathbf{C}$, $1 \leqslant j \leqslant d$, and $h \in D(T_0)$ such that

$$g = \sum_{r=1}^{d} \alpha_r \beta_r + h,$$

which implies that

$$[f:g]=\left[f:\sum_{r=1}^{d}\alpha_r\beta_r+h\right]=\sum_{r=1}^{d}\bar{\alpha}_r[f:\beta_r]=0.$$

Hence, condition (2) in Proposition 3.3 holds.

Now, suppose $[f:g]=0$ for $f\in D(T_1)$ and for all $g\in D$. It follows that $[f:\beta_j]=0$, $1\leqslant j\leqslant d$, since $\beta_j\in D$. This implies $f\in D$. Hence, condition (3) in Proposition 3.3 holds. The proof is complete. □

The following result gives a close relationship between GKN-sets for $\{T_0,T_1\}$ and domains of self-adjoint extensions of T_0. It is a direct consequence of Propositions 3.5 and 3.6 and Lemma 3.1.

Theorem 3.1. *Assume that* (3.5) *holds.*

(1) *If $\{\beta_r\colon\ r=1,2,\dots,d\}$ is a GKN-set for $\{T_0,T_1\}$, then the operator $T:D(T)\to l_W^2(\mathbf{I})$ defined by*

$$D(T):=D,\quad Tf:=T_1f,\quad f\in D(T) \tag{3.15}$$

is a self-adjoint extension of T_0, where D is defined by (3.12).

(2) *Conversely, if T is a self-adjoint extension of T_0, then there exists a GKN-set $\{\beta_j\}_{j=1}^d$ such that T is determined by* (3.15).

3.4. Relationships between self-adjoint extensions and Lagrangian subspaces

In this subsection, we establish one-to-one correspondences among the set of all the self-adjoint extensions of T_0, the set of all the d-dimensional Lagrangian subspaces of the boundary space, and the set of all the complete Lagrangian subspaces of the boundary space S.

Denote

$$\begin{aligned}
&p:=\max\bigl\{\text{complex dimension of subspaces of } S \text{ where on } \operatorname{Im}[f:f]>0\bigr\},\\
&q:=\max\bigl\{\text{complex dimension of subspaces of } S \text{ where on } \operatorname{Im}[f:f]<0\bigr\},\\
&Ex:=p-q,\\
&\Delta:=\max\{\text{complex dimension of Lagrangian subspaces of } S\}.
\end{aligned}$$

The integers p and q are called the positivity index and the negativity index of S, respectively, Ex is called the excess of positivity over negativity indices, and Δ is called the Lagrangian index of S [13, Definition 1, pp. 28–29].

Lemma 3.3. *$p=d^+$, $q=d^-$, and $Ex=d^+-d^-$. Consequently, $d^+=d^-=d$ if and only if $Ex=0$.*

Proof. This proposition can be easily verified by using the fact

$$[\hat f:\hat g]=\pm 2i(f,g)_W\quad \forall f,g\in D^\pm,\qquad [\hat f:\hat g]=0\quad \forall f\in D^+,\ \forall g\in D^-,$$

and by (4) in Proposition 3.2. So the details of the proof are omitted. □

It follows from Lemma 3.3 and Theorem 1 in [13, p. 29] that

$$\dim S = d^+ + d^- = p + q, \qquad \Delta = \min\{p, q\} = \bigl(\dim S - |Ex|\bigr)/2. \tag{3.16}$$

So, by Lemma 3.3 and from (3.16), one can get the following result.

Proposition 3.7.

(1) *T_0 has a self-adjoint extension if and only if $Ex = 0$.*
(2) *Assume that condition* (3.4) *holds. Then S has a d-dimensional Lagrangian subspace if and only if $Ex = 0$.*

Lemma 3.4. *Assume that condition* (3.4) *holds. Then L is a d-dimensional Lagrangian subspace of S if and only if L is a complete Lagrangian subspace of S.*

Proof. First consider the necessity. Suppose that L is a d-dimensional Lagrangian subspace of S with a orthonormal basis $\{\hat{f}^j\}_{j=1}^d$. It is evident that $\{f^j\}_{j=1}^d$ is a GKN-set. By Proposition 3.7, T_0 has a self-adjoint extension T. It follows from Theorem 3.1 that

$$D(T) = \bigl\{f\colon \bigl[f : f^j\bigr] = 0, \quad 1 \leqslant j \leqslant d\bigr\}$$

is the domain of T. Further, from Lemma 3.2, we have $L = D(T)/D(T_0)$. Now we show that L is complete. For any $\hat{h} \in S$ satisfying

$$[\hat{h} : \hat{g}] = 0 \quad \text{for all } \hat{g} \in L,$$

we have

$$(T_1 h, g)_W = (h, T_1 g)_W = (h, Tg)_W \quad \text{for all } g \in D(T),$$

which implies that $h \in D(T^*) = D(T)$, and consequently, $\hat{h} \in L$. Therefore, L is a complete Lagrangian subspace of S.

Next, we consider the sufficiency. Suppose that L is a complete Lagrangian subspace of S. Define

$$D(T) = \psi^{-1} L = \bigl\{h \in D(T_1)\colon \hat{h} \in L\bigr\}, \quad T = T_1|_{D(T)},$$

where ψ is defined by (3.2). It is evident that $D(T)$ is a linear subspace of $l_W^2(I)$ and

$$D(T_0) \subset D(T) \subset D(T_1).$$

It is to show that T is self-adjoint. Since L is Lagrangian, we have

$$(Tf, g)_W - (f, Tg)_W = [f : g] = [\hat{f} : \hat{g}] = 0 \quad \forall f, g \in D(T),$$

which yields that $D(T) \subset D(T^*)$. On the other hand, for any $f \in D(T^*)$, we have

$$(T^* f, g)_W = (f, Tg)_W \quad \forall g \in D(T),$$

which implies that

$$[\hat{f} : \hat{g}] = [f : g] = (T_1 f, g)_W - (f, T_1 g)_W = 0.$$

So, $[\hat{f} : L] = 0$. From the assumption that L is complete, it follows that $\hat{f} \in L$. Hence, $f \in D(T)$ and consequently, $D(T^*) \subset D(T)$. Thus $D(T) = D(T^*)$ and T is a self-adjoint extension of T_0.

By Theorem 3.1, there exists a GKN-set $\{\beta_j\}_{j=1}^d$ such that T is determined by (3.15) with $D(T)$ defined by (3.12). From Lemma 3.2, we have

$$L = D(T)/D(T_0) = \operatorname{span}\{\hat{\beta}^1, \dots, \hat{\beta}^d\}.$$

Therefore, L is a d-dimensional Lagrangian subspace of S. The proof is complete. □

Proposition 3.8. *Assume that* (3.5) *holds.*

(1) *If T is a self-adjoint extension of T_0, then $L = D(T)/D(T_0)$ is a d-dimensional Lagrangian subspace of S.*
(2) *If L is a d-dimensional Lagrangian subspace of S, then the operator T as the restriction of T_1 to the domain*

$$D(T) = \psi^{-1}L = \{f \in D(T_1):\ \hat{f} \in L\}$$

is a self-adjoint extension of T_0.

Proof. By Lemma 3.4 and from the last part of the proof of Lemma 3.4, this proposition can be easily proved. So the details of the proof are omitted. □

Theorem 3.2. *Assume that condition* (3.4) *holds.*

(1) *T_0 has a self-adjoint extension T with domain $D(T)$ if and only if S has a d-dimensional Lagrangian subspace L. Furthermore,*

$$L = D(T)/D(T_0), \qquad D(T) = \psi^{-1}L. \tag{3.17}$$

(2) *There exists a natural one-to-one correspondence between the set $\mathcal{T}$ and the set $\mathcal{L}$, where $\mathcal{T}$ consists of all the self-adjoint extensions of T_0 and the set $\mathcal{L}$ consists of all the d-dimensional Lagrangian subspace of S.*

Proof. Result (1) can be easily verified by Propositions 3.7 and 3.8.

To show result (2), we define a map $\varphi : \mathcal{T} \to \mathcal{L}$ by $\varphi T = L = D(T)/D(T_0)$. The map φ is well defined and surjective from result (1). Suppose that T_α and T_β are any two different self-adjoint extensions of T_0. Then $D(T_\alpha) \neq D(T_\beta)$. Suppose that there exists an element $u \in D(T_\alpha)$, but $u \notin D(T_\beta)$. Denote

$$L_\alpha := \varphi T_\alpha = D(T_\alpha)/D(T_0), \qquad L_\beta := \varphi T_\beta = D(T_\beta)/D(T_0).$$

It is evident that $\hat{u} = \psi u = \{u + D(T_0)\} \in L_\alpha$, but $\hat{u} \notin L_\beta$. This implies that $L_\alpha \neq L_\beta$ and consequently, φ is injective. Hence, φ is a bijective map. Result (2) is shown. This completes the proof. □

The following result is a direct consequence of Lemma 3.4 and Theorem 3.2.

Theorem 3.3. *Assume that condition* (3.4) *holds.*

(1) *T_0 has a self-adjoint extension T if and only if S has a complete Lagrangian subspace L. Furthermore, $D(T)$ and L satisfy* (3.17).

(2) *There exists a natural one-to-one correspondence between the set $\mathcal{T}$ and the set $\mathcal{L}$, where $\mathcal{T}$ consists of all the self-adjoint extensions of T_0 and the set $\mathcal{L}$ consists of all the complete Lagrangian subspace of S.*

Remark 3.1. Theorems 3.2 and 3.3 are discrete analogs of the GKN–EZ Theorem in [13], in which Everitt and Markus gave the similar results for quasi-differential expressions.

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